where $\Gamma(z)$ is the gamma function. One sees that the antisymmetric field (l_1) corresponds to terms decreasing roughly as p^{-3+t}, while the symmetric part (l_2) corresponds to terms as p^{-1-t}. As (B4) gives t near unity, especially for low ε, one cannot neglect the antisymmetric part. We retained the two kinds of terms in our calculation.

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Improved Accuracy for Commensurate-Line Synthesis

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***Abstract*—By employing a simple transformation that preserves numerical accuracy, improved precision is obtainable using a Richards' extraction technique to obtain characteristic impedances of commensurate transmission-line structures. Furthermore, reduced sensitivity to coefficient truncation can result in computational savings.**

In this short paper, we would like to report on certain aspects of the numerical calculation of characteristic impedances of cascaded commensurate-line networks. If correctly employed, Richards' extraction can be an extremely simple yet powerful and well-behaved algorithm. If misapplied, it can create large numerical inaccuracies.

The most important application of commensurate-line synthesis has been to problems of insertion loss design. The prescribed problem is typically the realization of a transducer gain function

$$s_{21}(\lambda)s_{21}(-\lambda) = \left.\frac{(1-\lambda^2)^n}{P_n(-\lambda^2)}\right|_{\lambda=j\Omega} = |s_{21}(j\Omega)|^2 \qquad (1)$$

where P_n is an even polynomial of degree $2n$ and the transformed frequency variable is

$$\Sigma + j\Omega = \lambda = \tanh p\tau, \qquad p = \sigma + j\omega$$

where ω is radian frequency.[1] For example Levy [1] gives commensurate-line characteristic impedances for low-pass Chebychev filter transducer gain functions. We here consider the numerical synthesis of arbitrary (but realizable) transducer gain functions, for example low-pass or bandpass filters, broad-band transformers, delay lines with amplitude selectivity, equalizers, matching networks, etc.

Considering a lossless reciprocal 2 port in the λ domain, the unitary requirement demands that the resistively terminated 2 port have an input reflection factor $s_{11}(\lambda)$ satisfying

$$s_{11}(\lambda)s_{11}(-\lambda) = 1 - s_{21}(\lambda)s_{21}(-\lambda)|_{\lambda=j\Omega} = |s_{11}(j\Omega)|^2.$$

The function $s_{11}(\lambda)$ is found by choosing the appropriate numerator and denominator root factors. The denominator must be Hurwitz but there is generally considerable flexibility in the choice of numerator roots, as well as a choice of a $\pm$ sign. Our task is to consider a suitable numerical method to determine the characteristic impedances of the structure, once $s_{11}(\lambda)$ has been given. We can of course proceed directly to the use of Richards' theorem for extracting the lines, given $s_{11}(\lambda)$ the input reflection factor of the resistively terminated cascade. However, this can lead to large numerical errors.

Our technique is to numerically operate on the input reflection factor $s_{11}(\lambda)$ of the resistively terminated cascade in a manner that preserves numerical accuracy and yields the reflection factor of the cascade of lines terminated in a short or open circuit rather than in a resistance. We have found that synthesizing this lossless function by Richards' theorem is superior numerically to making the line extraction calculations directly on $s_{11}(\lambda)$. In the latter case we deal with a two-element-kind network, whose input impedance is complex at real frequencies. If we use the lossless reflection factor, we deal with a purely reactive unit element network.

Separate the numerator and denominator terms of $s_{11}(\lambda)$ into even and odd parts

$$s_{11}(\lambda) = \frac{h_e(\lambda) + h_o(\lambda)}{g_e(\lambda) + g_o(\lambda)}. \qquad (2)$$

The reflection factors of the unterminated cascade are then

$$s_{1s} = \frac{(g_o + h_o) - (g_e - h_e)}{(g_o + h_o) + (g_e - h_e)} \qquad (3)$$

$$s_{1o} = \frac{(g_e + h_e) - (g_o - h_o)}{(g_e + h_e) + (g_o - h_o)} \qquad (4)$$

$$s_{2s} = \frac{(g_o + h_o) - (g_e + h_e)}{(g_o + h_o) + (g_e + h_e)} \qquad (5)$$

$$s_{2o} = \frac{(g_e - h_e) - (g_o - h_o)}{(g_e - h_e) + (g_o - h_o)}. \qquad (6)$$

Here s_{1s} and s_{1o} are the reflection factors at port 1 when the opposite port is short or open circuited, respectively, and s_{2s} and s_{2o} are similarly defined at port 2. These relations are generally valid starting with any resistively terminated reactance 2 port, e.g., interdigital filters, but must be slightly modified if $s_{12}(0) =$

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[1] $\lambda = \coth p\tau$ may be used equally well.

0. The latter case does not arise when the 2-port is a cascade of unit elements. Coefficient subtractions, which can lead to reduced numerical accuracy may be completely avoided in four specific cases and then the choice of which of the four above reflection factors to synthesize by Richards' extractions is clear. We note that since the denominator of $s_{11}(\lambda)$ is Hurwitz, g_e and g_o are polynomials with positive coefficients. If the numerator polynomial of $s_{11}(\lambda)s_{11}(-\lambda)$ is factored to put all the roots of $s_{11}(\lambda)$ in the left-half plane (LHP), then the numerator polynomial has all its coefficients of the same sign. If the factorization is chosen to put all the numerator roots in the right-half plane (RHP), the coefficients of descending powers of λ alternate in sign. Hence to avoid subtractions in these cases, use the following table. There are four cases because of the $(\pm)$ sign.

Location of Zeros of $s_{11}(\lambda)$	Sign of Coefficients in $s_{11}(\lambda)$ h_e	h_o	Use Function
LHP	+	+	s_{2s}
RHP	+	−	s_{1o}
RHP	−	+	s_{1s}
LHP	−	−	s_{2o}

If a numerator factorization of $s_{11}(\lambda)$ is chosen which has both left- and right-half plane roots, we choose the reflection factor which gives the least number of subtractions and/or avoids subtractions of coefficients whose magnitudes are close numerically.

In the process of synthesis of any of the lossless reflection factors by Richards' theorem we *should* get cancellation of exact $(1 \pm \lambda)$ factors. Numerically, we simply delete the remainder errors, i.e., assume exact cancellation at each step, and term this process in the program "remainder truncation." Employing a straightforward Richards' extraction [2] on the appropriate function of (3), (4), (5), or (6) with remainder truncation, we arrive at a cascaded line realization. The program is simple enough that a routine for extracting up to five lines has been written for the HP-25 calculator.

As a yardstick of performance gains that can be expected by utilizing this transformation, the matrix reduction technique proposed by Youla *et al.* in [4] was used on $s_{11}(\lambda)$. This avoids the matrix inversion procedure necessary in [3].

Tables I–III show typical results obtained on a wide variety of data for a series of ten line prototype filters. On well-behaved wide-band data as in Table I our simple technique of using Richards' theorem on the appropriate reflection factor (s_{1o}) of the lossless system produced similar accuracy when compared to the matrix factorization technique applied to $s_{11}(\lambda)$, both methods employing double precision arithmetic. Reduced sensitivity to coefficient truncation for this well-behaved data allowed use of single precision arithmetic in our Richards' extraction process on $s_{1o}(\lambda)$ with a maximum error of 4.1 percent in line 10. On ill-conditioned, narrow-band data the improvement in using Richards' theorem on the appropriate lossless reflection factor versus matrix factorization applied to $s_{11}(\lambda)$ was remarkable. On these examples the modified Richards' technique was clearly superior by a significant margin (Tables II, III).

The simple Richards' procedure proposed here allows us to expect a minimum of six digit accuracy on narrow-band data and ten digit accuracy on wide-band data. The more complicated

TABLE I
WIDE-BAND DATA (LOW-PASS FILTER)

Low Pass Filter

$Z_G = 1.0$ $Z_L = 2.9779$

$S_{11}(\lambda)$ Coefficients[b]

	λ^{10}	λ^9	λ^8	λ^7	λ^6	λ^5	λ^4	λ^3	λ^2	λ	λ^0
Numerator	121.7	-136.2	209.2	-151.8	111.8	-52.45	21.75	-6.072	1.298	-.165	.0105
Denominator	121.7	167.9	248.8	206.7	152.6	79.44	33.76	10.31	2.29	.316	.0211

NORMALIZED ERROR

Extracted Line	Matrix Reduction Double Precision (Ref. [4])	Modified Data Richards' Extraction[a] Double Precision	Modified Data Richards' Extraction[a] Single Precision	Normalized Nominal Characteristic Impedances
1	0	$-.17 \times 10^{-15}$	$-.69 \times 10^{-6}$	1.26
2	$-.20 \times 10^{-15}$	$-.98 \times 10^{-16}$	$-.31 \times 10^{-5}$	0.566
3	$-.17 \times 10^{-13}$	$.22 \times 10^{-14}$	$-.27 \times 10^{-4}$	2.33
4	$.17 \times 10^{-12}$	$.34 \times 10^{-13}$	$-.14 \times 10^{-3}$	0.387
5	$.14 \times 10^{-10}$	$.15 \times 10^{-12}$	$-.50 \times 10^{-3}$	2.79
6	$-.57 \times 10^{-10}$	$.65 \times 10^{-12}$	$-.10 \times 10^{-2}$	0.355
7	$-.55 \times 10^{-10}$	$-.25 \times 10^{-13}$	$-.13 \times 10^{-2}$	2.92
8	$-.45 \times 10^{-10}$	$.99 \times 10^{-11}$	$-.13 \times 10^{-2}$	0.346
9	$.14 \times 10^{-9}$	$-.73 \times 10^{-10}$	$-.13 \times 10^{-2}$	2.97
10	$.87 \times 10^{-10}$	$.64 \times 10^{-9}$	$-.13 \times 10^{-2}$	0.343

[a] Uses s_{1o}, open circuit reflection factor of front end.
[b] Actual data, 16 decimal places. The coefficients of λ^{10} differ slightly.

TABLE II
NARROW-BAND DATA, SYMMETRIC CASE (BAND ELIMINATION FILTER)[a]

$Z_G = 1.0$ $Z_L = 1.0$

$S_{11}(\lambda)$ Coefficients[c]

	λ^{10}	λ^9	λ^8	λ^7	λ^6	λ^5	λ^4	λ^3	λ^2	λ	λ^0
Numerator	$-.3 \times 10^{-14}$	665.3	0.0	31.5	-3×10^{-14}	.5096	$-.1 \times 10^{-15}$	2.75×10^{-3}	$-.1 \times 10^{-17}$	1.3×10^{-5}	$.15 \times 10^{-20}$
Denominator	$.15 \times 10^{-15}$	665.3	257.8	81.5	16.4	2.61	.308	.028	1.76×10^{-3}	7.3×10^{-5}	$.15 \times 10^{-5}$

NORMALIZED ERROR

Extracted Line	Matrix Reduction Double Precision (Ref. [4])	Modified Data Richards' Extraction[b] Double Precision	Normalized Nominal Characteristic Impedances
1	$.42 \times 10^{-16}$	$.13 \times 10^{-15}$	5.27
2	$-.52 \times 10^{-13}$	$.16 \times 10^{-13}$	0.111
3	$.34 \times 10^{-10}$	$.25 \times 10^{-11}$	11.5
4	$-.12 \times 10^{-7}$	$.17 \times 10^{-9}$	0.091
5	$.21 \times 10^{-5}$	$.65 \times 10^{-8}$	12.3
6	$-.63 \times 10^{-5}$	$.84 \times 10^{-8}$	12.3
7	$-.36 \times 10^{-3}$	$-.37 \times 10^{-6}$	0.091
8	$.56 \times 10^{-2}$	$.41 \times 10^{-6}$	11.5
9	$.92 \times 10^{-1}$	$-.62 \times 10^{-6}$	0.111
10	.11	$-.15 \times 10^{-5}$	5.27

[a] Courtesy of R. Levy of MDL.
[b] Uses $s_{2s}(\lambda)$, short circuit reflection factor of back end.
[c] Actual data, 16 decimal places. The coefficients of λ^9 differ slightly.

TABLE III
NARROW-BAND DATA, ANTIMETRIC CASE (LOW-PASS FILTER)[c]

$Z_G = 1.0$ $Z_L = 1.0$

$S_{11}(\lambda)$ Coefficients[d]

	λ^{10}	λ^9	λ^8	λ^7	λ^6	λ^5	λ^4	λ^3	λ^2	λ	λ^0
Numerator	656.05	$.6 \times 10^{-13}$	40.45	7×10^{-14}	.873	$-.3 \times 10^{-15}$	7.66×10^{-3}	$-.8 \times 10^{-17}$	2.37×10^{-5}	$-.2 \times 10^{-19}$	0.0
Denominator	656.05	254.2	89.71	19.78	3.65	.506	5.64×10^{-2}	4.74×10^{-3}	2.94×10^{-4}	1.19×10^{-5}	2.4×10^{-7}

NORMALIZED ERROR[b]

Extracted Line	Modified Data Richards' Extraction[a] Double Precision	Normalized Nominal Characteristic Impedances
1	$.73 \times 10^{-15}$	0.190
2	$.17 \times 10^{-13}$	9.04
3	$.35 \times 10^{-11}$	0.087
4	$.46 \times 10^{-9}$	11.0
5	$.32 \times 10^{-7}$	0.081
6	$.10 \times 10^{-5}$	12.3
7	$.12 \times 10^{-5}$	0.091
8	$.21 \times 10^{-5}$	11.5
9	$.98 \times 10^{-6}$	0.111
10	$-.17 \times 10^{-6}$	5.27

[a] Uses $s_{2s}(\lambda)$, short circuit reflection factor of back end.
[b] For this case, matrix reduction, double precision terminated while factoring the resistivity matrix because the square root of a negative number was required due to large errors.
[c] Courtesy of R. Levy of MDL.
[d] Actual data, 16 decimal places. The coefficients of λ^{10} differ slightly.

programming of the matrix triangulation approach [4] could also be applied to the transformed lossless reflection factor functions yielding even greater accuracy. However, it appears that in practice the simple Richards' algorithm with remainder truncation applied to the appropriate lossless reflection factor data gives excellent results.

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Differential Phase Shift at Microwave Frequencies Using Planar Ferrites

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Abstract—**It is generally believed that planar ferrites are not useful in differential-phase-shift devices operated below resonance. A configuration of planar ferrite is suggested, which leads to appreciable differential phase shift on application of very small magnetic fields.**

The planar ferrite has an easy plane of magnetization. A small external magnetic field directed along some axis (z axis) in this plane saturates the sample magnetically. The nonunit diagonal components μ_{xx} and μ_{yy} of the permeability tensor are unequal [1]. The planar ferrites, when used in devices operating in Q and K bands, need the application of very small magnetic fields.

Bady [1, p. 59] stated that "... planar ferrites are not as desirable as isotropic ferrites ..." in operation below resonance because the change in susceptibility is small. In his case the easy plane yz (see Fig. 1) was in the plane of the slab and contained the direction of propagation y. It is shown in the following discussion that much higher differential phase shift can be achieved by orienting the easy plane normal to the direction of propagation.

Consider a thin slab of transversely magnetized ferrite, kept in a waveguide supporting only the dominant mode propagating along the y axis. The direction of magnetization is along the z axis. The broad face of the slab is normal to the x axis. The width of the guide along x axis is L, the distance of the slab from the sidewall at $x = 0$ is a and the slab thickness is δ. With μ_{xx} not necessarily the same as μ_{yy}, an approximate expression

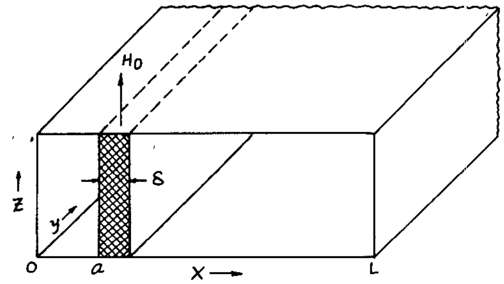

Fig. 1. A single planar ferrite slab in a rectangular waveguide of internal width L.

for the differential phase shift can be obtained [2] as follows:

$$\beta_+ - \beta_- = -\frac{2\pi\delta}{L^2}\cdot\frac{K}{\mu_{xx}}\sin 2\pi a/L \quad (1)$$

where β_+ and β_- are the propagation constants for the forward and the reverse propagation, K is the modulus of the off-diagonal element of the permeability tensor, and $\delta/L \simeq 0.01$ or so.

For a differential phase shifter using an isotropic ferrite slab (case 1), $\mu_{xx} = \mu_{yy}$ and (1) takes the familiar form [2]

$$\beta_+ - \beta_- = -\frac{2\pi\delta}{L^2}\cdot\frac{\omega\cdot 4\pi M_0/\gamma}{H_0(H_0 + 4\pi M_0) - \omega^2/\gamma^2}\sin 2\pi a/L \quad (2)$$

where ω, M_0, and H_0 are the operating frequency, saturation magnetization, and the applied magnetic field, respectively. γ is the gyromagnetic ratio to be taken as 1.76×10^7 rad/s.

For a differential phase shifter with the plane yz as the easy plane (case 2)

$$\beta_+ - \beta_- = -\frac{2\pi\delta}{L^2}\cdot\frac{\omega\cdot 4\pi M_0/\gamma}{H_0(H_0 + H_a + 4\pi M_0) - \omega^2/\gamma^2}\sin 2\pi a/L \quad (3)$$

where H_a is the anisotropy field and H_0 is the applied field, not necessarily the same as that in (2).

The ferrite phase shifters are normally operated below resonance to avoid attenuation. When the applied field is small, the first term in the denominators of (2) and (3) can be neglected in comparison with the second, and thus insignificant differential phase shifts result in both cases at relatively high frequencies.

The performance of a phase shifter using planar ferrite can be improved very much by orienting the easy plane normal to the direction of propagation (case 3). The expression for the differential phase shift assumes the following form:

$$\beta_+ - \beta_- = -\frac{2\pi\delta}{L^2}\cdot\frac{\omega\cdot 4\pi M_0/\gamma}{(H_0 + 4\pi M_0)(H_0 + H_a) - \omega^2/\gamma^2}\sin 2\pi a/L. \quad (4)$$

When M_0 and H_a are chosen suitably, considerably large differential phase shift can be obtained even when H_0 is small.

Normalized differential phase shift for the three aforementioned cases has been plotted in Fig. 2. It is obvious that maximum differential phase shift is obtainable in case 3 for small magnetic fields. However, in case 3 one would like to avoid the region $H_0 > 0.5$ kOe to keep away from resonance.

One disadvantage of the configuration used in case 3 is an overdependence of differential phase shift on the material

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